the students on the amount of detail they supply and on how well they write up these problems. Some earlier possibilities are Exercises 2,3, or 8 of Section 3.4 and Exercises 2 or 6 of Section 3.5.

3.1 Convergence

1. Let $\varepsilon > 0$. (When doing these for students, we suggest working out the last step first in order to decide how to choose n_0.)

   (a) By the Archimedean Principle $\exists\ n_0 \in \mathbb{N}$ such that $\dfrac{1}{n_0} < \sqrt{\varepsilon}$. Then $n \geq n_0 \Rightarrow$

   $$\left|\frac{1}{n^2} - 0\right| = \frac{1}{n^2} \leq \frac{1}{n_0^2} < \varepsilon.$$

   (b) Choose $n_0 \in \mathbb{N}$ with $\dfrac{1}{n_0} < \dfrac{\varepsilon}{6}$. Then $n \geq n_0 \Rightarrow$

   $$\left|\frac{3n}{n+2} - 3\right| = \frac{6}{n+2} < \frac{6}{n} \leq \frac{6}{n_0} < \varepsilon.$$

   (c) Choose $n_0 \in \mathbb{N}$ with $\frac{1}{n_0} < \frac{25}{29}\varepsilon$. Then $n \geq n_0 \Rightarrow$

   $$\left|\frac{3n+7}{5n+2} - \frac{3}{5}\right| = \frac{29}{25n+10} < \frac{29}{25n} \leq \frac{29}{25} \cdot \frac{1}{n_0} < \varepsilon.$$

   (d) Choose $n_0 \in \mathbb{N}$ such that $\dfrac{1}{n_0} < \dfrac{2}{\sqrt{3}}\sqrt{\varepsilon}$. Then $n \geq n_0 \Rightarrow$

   $$\left|\frac{n^2+1}{2n^2+5} - \frac{1}{2}\right| = \frac{3}{4n^2+10} < \frac{3}{4n^2} \leq \frac{3}{4} \cdot \frac{1}{n_0^2} < \frac{3}{4} \cdot \frac{4\varepsilon}{3} = \varepsilon.$$

   (e) Choose $n_0 \in \mathbb{N}$ with $\dfrac{1}{n_0} < \varepsilon$. Then $n \geq n_0 \Rightarrow$

   $$\left|\frac{\sin n}{n} - 0\right| = \frac{|\sin n|}{n} \leq \frac{1}{n} \leq \frac{1}{n_0} < \varepsilon.$$

   (f) Choose $n_0 \in \mathbb{N}$ such that $\dfrac{1}{n_0} < \varepsilon$. Then $n \geq n_0 \Rightarrow$

   $$\left|\frac{-n}{n^2+1} - 0\right| = \frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n} \leq \frac{1}{n_0} < \varepsilon.$$

2. $\lim_{n\to\infty} x_n = 0$ since $|x_n| \leq \frac{1}{n}\ \forall\, n \in \mathbb{N}$.

3. $\lim_{n\to\infty} x_n$ does not exist. $(x_n)_{n\in\mathbb{N}} = \left(1, 1, \frac{1}{3}, 1, \frac{1}{5}, 1, \frac{1}{7}, 1, \ldots\right)$ is frequently outside of $\left(\frac{1}{2}, \frac{3}{2}\right)$, and so $(x_n)_{n\in\mathbb{N}}$ does not converge to 1. For $x \neq 1$, let U be any neighborhood of x not containing 1. Then $(x_n)_{n\in\mathbb{N}}$ is frequently outside of U and hence does not converge to x. (Of course, it would be easier here to use subsequences, but we do not cover them until Section 3.3.)

4. (a) Let $\varepsilon > 0$. Since $x_n \to x$, $\exists\, n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow |x_n - x| < \varepsilon$. By Corollary 2.1, $||x_n| - |x|| \leq |x_n - x| < \varepsilon$ for $n \geq n_0$. Therefore, $|x_n| \to |x|$.
   (b) Let $\varepsilon > 0$. Since $|x_n| \to 0$, $\exists\, n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow |x_n| = ||x_n| - 0| < \varepsilon$. Then $n \geq n_0 \Rightarrow |x_n - 0| = |x_n| < \varepsilon$, and so $x_n \to 0$.
   (c) Let $(x_n)_{n\in\mathbb{N}} = (-1, 1, -1, 1, -1, 1, \ldots)$.

5. First note that if $x < 0$, then $(x_n)_{n\in\mathbb{N}}$ would eventually be in $\left(\frac{3}{2}x, \frac{1}{2}x\right)$ and hence $(x_n)_{n\in\mathbb{N}}$ would eventually be negative. Also, see Theorem 3.3 in Section 3.2.

Case 1: $x = 0$. For $\varepsilon > 0$, choose $n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow x_n < \varepsilon^2$. Then $\left|\sqrt{x_n} - 0\right| = \sqrt{x_n} < \varepsilon$ for $n \geq n_0$., and so $\sqrt{x_n} \to 0$.

Case 2: $x > 0$. For $\varepsilon > 0$, choose $n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow |x_n - x| < \varepsilon\sqrt{x}$. Then

$$\begin{aligned}
\left|\sqrt{x_n} - \sqrt{x}\right| &= \left|\left(\sqrt{x_n} - \sqrt{x}\right) \cdot \frac{\sqrt{x_n} + \sqrt{x}}{\sqrt{x_n} + \sqrt{x}}\right| \\
&= \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \\
&\leq \frac{|x_n - x|}{\sqrt{x}} \\
&< \frac{\varepsilon\sqrt{x}}{\sqrt{x}} = \varepsilon
\end{aligned}$$

for $n \geq n_0$. Therefore, $\sqrt{x_n} \to \sqrt{x}$.

6. This is Proposition 3.2 in Section 3.2. We believe that students will appreciate the argument in Proposition 3.2 more for having first tried it themselves, whether or not they are successful.

7. Both sequences are unbounded. See Exercise 6.

8. if $r < 0$, then $|r^n| = |r|^n \to 0$ by Example 3.5. By Exercise 4(b), $r^n \to 0$.

9. (a) Since $0 < \frac{e}{\pi} < 1$, $\left(\frac{e}{\pi}\right)^n \to 0$ by Example 3.5.

   (b) $c^{\frac{1}{n}} = e^{\frac{1}{n}\ln c} \to e^0 = 1$.

   (c) $n^{\frac{1}{n}} = e^{\frac{1}{n}\ln n}$. Since $\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$ by L'Hôpital, $\lim_{n\to\infty} n^{\frac{1}{n}} = e^0 = 1$.

   (d) $\left(1+\frac{1}{n}\right)^{2n} = e^{2n\ln\left(1+\frac{1}{n}\right)}$. Since

$$\begin{aligned}
\lim_{n\to\infty} 2n\ln\left(1+\frac{1}{n}\right) &= \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{2n}} \\
&= \lim_{n\to\infty} \frac{\dfrac{\frac{d}{dn}\left(1+\frac{1}{n}\right)}{1+\frac{1}{n}}}{\frac{1}{2}\frac{d}{dn}\left(\frac{1}{n}\right)} \qquad \text{(by L'Hôpital)} \\
&= \lim_{n\to\infty} \frac{2}{1+\frac{1}{n}} = 2,
\end{aligned}$$

   $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{2n} = e^2$. The theorems in Section 3.2 justify the following argument: $\left(1+\frac{1}{n}\right)^{2n} = \left[\left(1+\frac{1}{n}\right)^n\right]^2 \to e^2$ by Example 3.6.

   (e) For $n \geq 4$, $\frac{n^2}{n!} = \frac{n}{(n-1)(n-2)\cdots(3)(2)(1)} \leq \frac{n}{(n-1)(n-2)} <$

$\frac{n}{n^2-3n} = \frac{1}{n-3}$. Given $\varepsilon > 0$, by Theorem 2.1, $\exists\, n_0 \in \mathbb{N}$ such that $n_0 > \frac{1}{\varepsilon} + 3$. Then $n \geq n_0 \Rightarrow n-3 \geq n_0 - 3 > \frac{1}{\varepsilon}$ or, equivalently, $n \geq n_0 \Rightarrow \frac{1}{n-3} < \varepsilon$. Therefore, $n \geq n_0 \Rightarrow 0 < \frac{n^2}{n!} < \frac{1}{n-3} < \varepsilon$, and so $\frac{n^2}{n!} \to 0$. (Theorem 3.4 of the next section will eliminate the latter part of this argument.)

(f) $\frac{n!}{n^n} = \left(\frac{1}{n}\right)\left(\frac{2}{n}\right)\left(\frac{3}{n}\right)\cdots\left(\frac{n-1}{n}\right) < \frac{1}{n}$ for $n \geq 3$. Since $\frac{1}{n} \to 0$, so does $\frac{n!}{n^n}$.

3.2 Limit Theorems

1. (a) $\frac{n}{n+2} = \frac{1}{1+\frac{2}{n}} \to \frac{1}{1+0} = 1$.

   (b) For n even, $\frac{(-1)^n n}{n+2} \to 1$ and for n odd, $\frac{(-1)^n n}{n+2} \to -1$. It follows that $\lim\limits_{n\to\infty} \frac{(-1)^n n}{n+2}$ does not exist.

   (c) $\frac{n^2+4n}{2n^2+5} = \frac{1+\frac{4}{n}}{2+\frac{5}{n^2}} \to \frac{1+0}{2+0} = \frac{1}{2}$.

   (d) $\left(3+\frac{1}{n}\right)^2 \to 3^2 = 9$.

   (e) $\sqrt{n} - \sqrt{n+1} = \left(\sqrt{n} - \sqrt{n+1}\right) \cdot \frac{\sqrt{n}+\sqrt{n+1}}{\sqrt{n}+\sqrt{n+1}} = \frac{-1}{\sqrt{n}+\sqrt{n+1}}$.
   So

$$0 \leq \left|\sqrt{n} - \sqrt{n+1}\right| = \frac{1}{\sqrt{n}+\sqrt{n+1}} < \frac{1}{\sqrt{n}} = \sqrt{\frac{1}{n}} \to \sqrt{0} = 0.$$

   Therefore, $\sqrt{n} - \sqrt{n+1} \to 0$ by the Squeeze Theorem and Exercise 3.1.4(b).

   (f)

$$n - \sqrt{n^2+n} \;=\; \sqrt{n}\left(\sqrt{n} - \sqrt{n+1}\right)$$

$$= \frac{-\sqrt{n}}{\sqrt{n}+\sqrt{n+1}} \quad \text{(by (e))}$$
$$= \frac{-1}{1+\sqrt{1+\frac{1}{n}}}$$
$$\to \frac{-1}{1+1} = -\frac{1}{2}.$$

2. Let $(x_n)_{n\in\mathbb{N}} = \left((-1)^{n+1}\right)_{n\in\mathbb{N}} = (1,-1,1,-1,\ldots)$ and let $(y_n)_{n\in\mathbb{N}} = ((-1)^n)_{n\in\mathbb{N}} = (-1,1,-1,1,\ldots)$. $\forall n \in N$, $x_n + y_n = 0$, $x_n \cdot y_n = -1 = \frac{x_n}{y_n}$.

3. Write $x_n = \dfrac{(x_n+y_n)+(x_n-y_n)}{2}$ and $y_n = \dfrac{(x_n+y_n)-(x_n-y_n)}{2}$. By Theorem 3.2 both $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ converge.

4. $(\frac{1}{n})_{n\in\mathbb{N}}$.

5. $\frac{-1}{n} \le \frac{\cos n}{n} \le \frac{1}{n} \forall n$. Since $\pm\dfrac{1}{n} \to 0$, $\dfrac{\cos n}{n} \to 0$ by the Squeeze Theorem.

6. Let $B > 0$ with $|x_n| \le B \;\forall\, n \in \mathbb{N}$. Then $0 \le |x_n y_n| \le B\,|y_n| \to 0$. By the Squeeze Theorem, $|x_n y_n| \to 0$ and so $x_n y_n \to 0$.

7. (a) Let $x_n = (-1)^n$ and $y_n = 1 \;\forall n \in \mathbb{N}$. Then $(x_n y_n)_{n\in\mathbb{N}} = (x_n)_{n\in\mathbb{N}}$ does not converge.

   (b) Let $x_n = n^2$ and $y_n = \frac{1}{n} \;\forall n \in \mathbb{N}$. Then $(x_n y_n)_{n\in\mathbb{N}} = (n)_{n\in\mathbb{N}}$ is unbounded and hence does not converge.

8. Let $x \in \mathbb{R}$. Since the irrational numbers are dense in $\mathbb{R}$, $\forall n \in \mathbb{N}$ choose an irrational $x_n \in (x-\frac{1}{n}, x+\frac{1}{n})$. Then $-\dfrac{1}{n} < x_n - x < \dfrac{1}{n} \;\forall n \in \mathbb{N}$. By the Squeeze Theorem, $x_n - x \to 0$ or, equivalently, $x_n \to x$.

3.3 Subsequences

1. (a) $(1,2,1,3,1,4,\ldots)$ has the constant sequence $(1,1,1,\ldots)$ as a subsequence.